# Canonical and Noncanonical Removal of Non-Resonant Terms: the Cubic Oscillator

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**Abstract**

We use a single nonlinear oscillator with a cubic Hamiltonian perturbation to illustrate the distinction between canonical and noncanonical perturbative changes of variables. A simple near-identity transformation in the physical variables can remove the cubic term from the Hamiltonian, but it does not preserve the symplectic form and therefore is not canonical. As a result, the full off-resonant expansion differs from the canonical normal form. Nevertheless, after projection onto the resonant manifold, the same quartic resonant coefficient is obtained. The example shows that noncanonical parametrizations can alter the off-resonant representation, while the resonant normal form contains the invariant dynamical content.

## 1 Canonical cubic oscillator

Consider the Hamiltonian

$$H(u,v) = \frac{1}{2}\left(v^2 + \omega^2 u^2\right) + \varepsilon \frac{\lambda}{3} u^3, \qquad \omega > 0. \tag{1}$$

The variables $(u, v)$ are canonical because the equations of motion have the standard Hamiltonian form

$$\dot{u} = \frac{\partial H}{\partial v}, \qquad \dot{v} = -\frac{\partial H}{\partial u}. \tag{2}$$

Equivalently,

$$\{u, v\} = 1, \qquad \Omega = du \wedge dv \tag{3}$$

is the canonical symplectic two-form.

Thus

$$\dot{u} = v, \qquad \dot{v} = -\omega^2 u - \varepsilon\lambda u^2, \tag{4}$$

or, equivalently,

$$\ddot{u} + \omega^2 u + \varepsilon\lambda u^2 = 0. \tag{5}$$

The unperturbed motion has one linear frequency $\omega$. At cubic order there is no exact resonance, because a cubic monomial carries frequency mismatch $\pm\omega$ or $\pm 3\omega$, but never zero. Therefore the cubic term is non-resonant.

### 1.1 Noncanonical removal in physical variables

We now remove the cubic term by a simple near-identity change of variables:

$$u = U - \varepsilon \frac{\lambda}{3\omega^2} U^2, \qquad v = V. \tag{6}$$

Substitution of (6) into (1) gives, through order $\varepsilon^2$,

$$H = \frac{1}{2}\left(V^2 + \omega^2 U^2\right) - \frac{5\varepsilon^2\lambda^2}{18\omega^2} U^4 + O(\varepsilon^3). \tag{7}$$

Thus the cubic term has disappeared. However, this removal was achieved by a change of variables that is not canonical.

Indeed,

$$du = \left(1 - \frac{2\varepsilon\lambda}{3\omega^2} U\right) dU, \qquad dv = dV. \tag{8}$$

Therefore

$$du \wedge dv = \left(1 - \frac{2\varepsilon\lambda}{3\omega^2} U\right) dU \wedge dV \neq dU \wedge dV. \tag{9}$$

Hence the transformation (6) does not preserve the symplectic form. It also does not preserve the phase-space area element:

$$du\, dv = \left(1 - \frac{2\varepsilon\lambda}{3\omega^2} U\right) dU\, dV. \tag{10}$$

Thus a small rectangle in the $(U, V)$ plane is mapped to a region in the original $(u, v)$ plane with a position-dependent area factor. The change of variables distorts phase space.

This is the essential point: the cubic term can be removed algebraically, but the Hamiltonian structure is not preserved in the standard canonical form $dU \wedge dV$.

### 1.2 Noncanonical complex representation

Define the complex amplitude associated with the variables $(U, V)$:

$$B = \sqrt{\frac{\omega}{2}}\, U + \frac{i}{\sqrt{2\omega}}\, V, \qquad \overline{B} = \sqrt{\frac{\omega}{2}}\, U - \frac{i}{\sqrt{2\omega}}\, V. \tag{11}$$

Then

$$U = \frac{B + \overline{B}}{\sqrt{2\omega}}, \qquad V = -i\sqrt{\frac{\omega}{2}}\left(B - \overline{B}\right). \tag{12}$$

Using (7), we obtain

$$H = \omega B\overline{B} - \frac{5\varepsilon^2\lambda^2}{72\omega^4}\left(B + \overline{B}\right)^4 + O(\varepsilon^3). \tag{13}$$

Expanding the quartic term,

$$H = \omega B\overline{B} - \frac{5\varepsilon^2\lambda^2}{72\omega^4}\left(B^4 + 4B^3\overline{B} + 6B^2\overline{B}^{\,2} + 4B\overline{B}^{\,3} + \overline{B}^{\,4}\right) + O(\varepsilon^3). \tag{14}$$

This expression contains both resonant and non-resonant quartic monomials.

For one oscillator, a monomial $B^m\overline{B}^{\,n}$ is resonant only if

$$m = n. \tag{15}$$

Therefore the only quartic resonant monomial is

$$B^2\overline{B}^{\,2} = |B|^4. \tag{16}$$

Projecting (14) onto the resonant part gives

$$H_{\rm res}^{(\rm noncan)} = -\frac{5\varepsilon^2\lambda^2}{72\omega^4}\cdot 6B^2\overline{B}^{\,2}. \tag{17}$$

Hence

$$H_{\rm res}^{(\rm noncan)} = -\frac{5\varepsilon^2\lambda^2}{12\omega^4}|B|^4. \tag{18}$$

## 2 Canonical resonant form in complex canonical variables

We now determine the canonical resonant Hamiltonian directly in terms of the canonical complex variables

$$b = \sqrt{\frac{\omega}{2}}\,u + \frac{i}{\sqrt{2\omega}}\,v, \qquad \bar{b} = \text{c.c.}, \tag{19}$$

which satisfy the canonical Poisson bracket

$$\{b, \bar{b}\} = -\,i. \tag{20}$$

In these variables, the quadratic Hamiltonian is

$$H_2 = \omega\, b\bar{b}. \tag{21}$$

Since the cubic term is non-resonant (single frequency system), the first resonant contribution appears at order $\varepsilon^2$. The canonical Hamiltonian must therefore take the normal-form structure

$$H_{\rm can}(b,\bar{b}) = \omega\, b\bar{b} + \alpha(b\bar{b})^2 + O(\varepsilon^3), \tag{22}$$

where $\alpha = O(\varepsilon^2)$.

### 2.1 Frequency determination

The frequency associated with (22) is obtained from Hamilton's equations:

$$\dot{b} = -i\frac{\partial H_{\rm can}}{\partial \bar{b}} = -i\left(\omega + 2\alpha\, b\bar{b}\right) b. \tag{23}$$

Thus, the nonlinear frequency is

$$\Omega = \omega + 2\alpha\,|b|^2. \tag{24}$$

On the other hand, from the original oscillator equation

$$\ddot{u} + \omega^2 u + \varepsilon\lambda u^2 = 0, \tag{25}$$

a standard Poincaré–Lindstedt expansion gives the amplitude-dependent frequency

$$\Omega(A) = \omega - \frac{5\varepsilon^2\lambda^2}{12\omega^3}A^2 + O(\varepsilon^3). \tag{26}$$

To express this result in terms of $b$, note that for the harmonic oscillator

$$u(t) \approx A\cos(\omega t) \quad \Longrightarrow \quad |b|^2 = \frac{1}{2}\,\omega A^2. \tag{27}$$

Hence

$$A^2 = \frac{2}{\omega}|b|^2. \tag{28}$$

Substituting into (26), we obtain

$$\Omega = \omega - \frac{5\varepsilon^2\lambda^2}{12\omega^3}\cdot\frac{2}{\omega}|b|^2 = \omega - \frac{5\varepsilon^2\lambda^2}{6\omega^4}|b|^2. \tag{29}$$

Comparing with (24), we identify

$$2\alpha = -\frac{5\varepsilon^2\lambda^2}{6\omega^4}, \qquad \Rightarrow \qquad \alpha = -\frac{5\varepsilon^2\lambda^2}{12\omega^4}. \tag{30}$$

### 2.2 Canonical resonant Hamiltonian

Substituting into (22), we obtain

$$H_{\text{can}} = \omega|b|^2 - \frac{5\varepsilon^2\lambda^2}{12\omega^4}|b|^4 + O(\varepsilon^3). \tag{31}$$

Thus, the resonant part of the canonical Hamiltonian is

$$H_{\text{res}}^{(\text{can})} = -\frac{5\varepsilon^2\lambda^2}{12\omega^4}|b|^4. \tag{32}$$

## 3 Comparison on the resonant manifold

The noncanonical calculation gives

$$H_{\text{res}}^{(\text{noncan})} = -\frac{5\varepsilon^2\lambda^2}{12\omega^4}|B|^4. \tag{33}$$

The canonical calculation gives

$$H_{\text{res}}^{(\text{can})} = -\frac{5\varepsilon^2\lambda^2}{12\omega^4}|b|^4. \tag{34}$$

The variables $B$ and $b$ differ away from resonance because $B$ was constructed after a noncanonical transformation, whereas $b$ is canonical. Therefore the full off-resonant expansions are different. However, the coefficient of the resonant quartic interaction is the same:

$$-\frac{5\varepsilon^2\lambda^2}{12\omega^4}. \tag{35}$$

This is the invariant content of the weakly nonlinear dynamics.

## 4 Relation between the noncanonical $B(U,V)$ and canonical $b(u,v)$

We now make explicit the relation between the noncanonical complex amplitude $B(U,V)$ and the canonical complex variable

$$b = \sqrt{\frac{\omega}{2}}\, u + \frac{i}{\sqrt{2\omega}}\, v, \qquad \bar{b} = \text{c.c.} \tag{36}$$

Using the noncanonical transformation

$$u = U - \varepsilon \frac{\lambda}{3\omega^2} U^2, \qquad v = V, \tag{37}$$

we substitute into $b(u,v)$:

$$b = \sqrt{\frac{\omega}{2}} \left( U - \varepsilon \frac{\lambda}{3\omega^2} U^2 \right) + \frac{i}{\sqrt{2\omega}} V. \tag{38}$$

Grouping terms,

$$b = \underbrace{\sqrt{\frac{\omega}{2}} U + \frac{i}{\sqrt{2\omega}} V}_{B} - \varepsilon \frac{\lambda}{3\omega^2} \sqrt{\frac{\omega}{2}} U^2. \tag{39}$$

Thus we obtain the relation

$$b = B - \varepsilon \frac{\lambda}{3\omega^2} \sqrt{\frac{\omega}{2}} U^2. \tag{40}$$

We now express $U$ in terms of $B, \bar{B}$:

$$U = \frac{B + \bar{B}}{\sqrt{2\omega}}. \tag{41}$$

Hence

$$U^2 = \frac{(B + \bar{B})^2}{2\omega}. \tag{42}$$

Substituting into (40), we obtain

$$b = B - \varepsilon \frac{\lambda}{6\omega^3} \sqrt{\frac{\omega}{2}}\, (B + \bar{B})^2. \tag{43}$$

### 4.1 Consequences

This relation shows that:

- $b \neq B$: the two complex amplitudes differ by a nonlinear correction,
- the correction contains both $B^2$ and $\bar{B}^2$ terms,
- therefore $B$ mixes positive and negative frequency components.

In contrast, $b$ is canonical and diagonalizes the linear Hamiltonian, while $B$ does not.

### 4.2 Loss of canonicity

Using the canonical Poisson bracket $\{u, v\} = 1$, one finds

$$\{b, \bar{b}\} = -i, \qquad \{B, \bar{B}\} = -i\left(1 - \frac{2\varepsilon\lambda}{3\omega^2}U\right). \tag{44}$$

Thus

$$\{B, \bar{B}\} \neq -i, \tag{45}$$

and the variables $(B, \bar{B})$ are not canonical.

This confirms that the nonlinear transformation $(u, v) \mapsto (U, V)$ induces a noncanonical complex representation.

### 4.3 Interpretation

The noncanonical variable $B$ differs from the canonical variable $b$ by a nonlinear mixing of modes. As a consequence:

- the Hamiltonian expressed in $B$ contains additional non-resonant terms,
- the representation depends on the choice of variables,
- but these differences affect only the off-resonant structure.

After projection onto the resonant manifold, both descriptions reduce to the same effective quartic interaction, as shown in the following section.

## 5 Conclusion

We have compared two distinct procedures for eliminating the non-resonant cubic term in a weakly nonlinear Hamiltonian system.

First, a *noncanonical* near-identity transformation,

$$u = U - \varepsilon\frac{\lambda}{3\omega^2}U^2, \qquad v = V, \tag{46}$$

removes the cubic contribution directly at the level of the Hamiltonian. However, this transformation does not preserve the symplectic structure:

$$du \wedge dv = \left(1 - \frac{2\varepsilon\lambda}{3\omega^2}U\right) dU \wedge dV \neq dU \wedge dV. \tag{47}$$

As a result, phase-space volume is distorted and the canonical structure of Hamilton's equations is lost. Consequently, the perturbation expansion written in the variables $(U, V)$ differs from the canonical one at higher orders, and contains additional non-resonant contributions.

In contrast, a *canonical* transformation preserves the symplectic form, and therefore maintains the Hamiltonian structure of the dynamics. In canonical complex variables $(b, \bar{b})$, the Hamiltonian can be reduced to a normal form depending only on the invariant quantity $|b|^2$. In particular, all non-resonant terms are systematically eliminated while preserving the underlying geometry.

Despite these differences, both approaches yield the same result when restricted to the resonant manifold. Projecting the dynamics onto the resonant sector (i.e., retaining only monomials satisfying $m = n$) gives in both cases the same effective Hamiltonian,

$$H_{\text{res}} = -\frac{5\varepsilon^2\lambda^2}{12\omega^4} I^2, \qquad I = |b|^2. \tag{48}$$

Thus, while the *full* perturbative expansions depend on the chosen variables and, in particular, on whether the transformation is canonical or not, the *resonant interaction coefficient* is invariant.

This example shows that:

- noncanonical transformations can simplify the Hamiltonian locally, but at the cost of altering the symplectic structure and introducing representation-dependent higher-order terms;
- canonical transformations preserve the phase-space geometry and isolate the intrinsic dynamics through a normal form;
- the resonant part of the Hamiltonian — obtained by projection onto the resonant manifold — is independent of the parametrization.

In this precise sense, the unique dynamical content of the system is encoded in the resonant normal form. All non-resonant contributions are removable and representation-dependent, whereas the resonant interaction coefficient is an invariant of the dynamics.